North Sydney Girls High School

2022 HSC TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 15)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

NAME: ______________________ TEACHER: ______________________

STUDENT NUMBER: | | | | | | | | |

Question	1-10	11	12	13	14	15	16	Total
Mark	/10	/14	/16	/15	/15	/14	/16	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1-10.

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1 What is a unit vector in the same direction as $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$?

A. $\frac{1}{21}\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

B. $\frac{1}{\sqrt{21}}\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$

C. $\frac{1}{14}\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

D. $\frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

2 Consider the statement "If you don't eat your meat then you won't get any pudding".

Which of the following is logically equivalent to this statement?

A. If you didn't get any pudding then you did not eat your meat.

B. If you ate your meat then you had pudding.

C. If you had pudding then you ate your meat.

D. If you had pudding then you did not eat your meat.

3 How many distinct roots does the equation $(z^4 - 1)(z^2 + 3iz - 2) = 0$ have in $\mathbb{C}$?

A. 3

B. 4

C. 5

D. 6

4 Given that $(x+iy)^{12}=a+bi$ where $\sqrt{x^2+y^2}=1$ and x, y, a, b are real, which of the following is equal to $(x-iy)^{-12}$?

A. $a+bi$

B. $a-bi$

C. $-a+bi$

D. $-a-bi$

5 A particle is undergoing simple harmonic motion centred at the origin. Its velocity, v metres per second, in terms of its displacement x metres, is given by the equation:

$$v^2=2\left(1-x^2\right).$$

Initially the particle is at the origin moving in the positive direction.

How long does it take to first reach its maximum displacement?

A. $\sqrt{2}\pi$ seconds

B. $\dfrac{\sqrt{2}\pi}{2}$ seconds

C. $\dfrac{\sqrt{2}\pi}{4}$ seconds

D. $\dfrac{\pi}{4\sqrt{2}}$ seconds

6 Consider the vectors $\overrightarrow{OA}=\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}2\\-1\\1\end{pmatrix}$ and $\overrightarrow{OC}=\begin{pmatrix}3\\\mu\\\lambda\end{pmatrix}$, where O is the origin.

What are the values of μ and λ if the points A, B and C are collinear?

A. $\mu=3$, $\lambda=2$

B. $\mu=-3$, $\lambda=2$

C. $\mu=3$, $\lambda=-2$

D. $\mu=-3$, $\lambda=-2$

7 Which of the following definite integrals has a positive value?

A. $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} x^3\left(1+\cos x\right) dx$

B. $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{\sin^{-1} x}{1+x^4}\right) dx$

C. $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{\cos^{-1} x}{e^{-x^2}}\right) dx$

D. $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \left(\frac{\tan^{-1} x}{e^{x}+e^{-x}}\right) dx$

8 Given that z is a complex number such that $\operatorname{Im}(z) \neq 0$, which of the following expressions must be equivalent to $\dfrac{4z\bar{z}}{(z-\bar{z})^2}$?

A. $1+\left(\dfrac{\operatorname{Re}(z)}{\operatorname{Im}(z)}\right)^2$

B. $-1-\left(\dfrac{\operatorname{Re}(z)}{\operatorname{Im}(z)}\right)^2$

C. $\dfrac{1}{(\operatorname{Im} z)^2}$

D. $-\dfrac{1}{(\operatorname{Im} z)^2}$

9 Given that P is the set of prime numbers, consider the following statement S.

S: "$\forall p \in P$ (p is of the form $4m+1 \Rightarrow p$ can be written as a sum of two squares)"

Which statement is equivalent to $\neg S$?

A. $\forall p \in P$, p is of the form $4m+1$ and p cannot be written as a sum of two squares.

B. $\exists p \in P$, p is not of the form $4m+1$ and p can be written as a sum of two squares.

C. $\forall p \in P$, p is not of the form $4m+1$ or p cannot be written as a sum of two squares.

D. $\exists p \in P$, p is of the form $4m+1$ and p cannot be written as a sum of two squares.

10 The complex number z is defined such that $|z+3-i| = 2$ and $\operatorname{Arg}(z) = \theta$.

What is the least positive value of θ ?

A. $\pi + \tan^{-1}\left(\frac{1}{3}\right) - \sin^{-1}\left(\frac{2}{\sqrt{10}}\right)$

B. $\pi + \tan^{-1}\left(-\frac{1}{3}\right) - \sin^{-1}\left(\frac{2}{\sqrt{10}}\right)$

C. $\tan^{-1}\left(-\frac{1}{3}\right) + \sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{10}}\right)$

D. $\tan^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{10}}\right)$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks) Use a SEPARATE writing booklet

(a) Let $z = 1 + 3i$ and $w = 2 - 6i$.

(i) Find $z + w$ in modulus-argument form. **2**

(ii) Write $\frac{z}{w}$ in the form $x + iy$, where x and y are real numbers. **2**

(b) Find the vector equation for the sphere $x^2 + 4x + y^2 - 6y + z^2 = 12$. **2**

(c) Find $\int \frac{x^2 - 2x}{x^2 - 2x + 10}\, dx$. **2**

(d) Use mathematical induction to prove that $4^n > 3n + 7$ for all integers $n > 1$. **3**

(e) A particle is moving in a straight line in simple harmonic motion. **3**

Its maximum speed is 4 m/s and its maximum acceleration is 12 m/s/s.

Find the amplitude and the period of the motion.

Question 12 (16 marks) Use a SEPARATE writing booklet

(a) Sketch the region on the Argand diagram where: **2**

$$\arg(z-2i)=\arg(z-3).$$

(b) Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^4 x \tan^2 x \, dx$. **3**

(c) Given that x is an integer, consider the statement below.

S: "If x^2-6x+5 is even then x is odd."

(i) Write down the contrapositive of this statement. **1**

(ii) Prove that the statement S is true. **2**

(d) Given that $z=e^{i\theta}$, solve $\left|z+\sqrt{2}\right|=1$, writing answers in exponential form. **3**

(e) Relative to a fixed origin O, the vector equations of two lines l_1 and l_2 are:

$$l_1:\ \underset{\sim}{r}=\begin{pmatrix}9\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}-8\\-3\\5\end{pmatrix} \quad \text{and} \quad l_2:\ \underset{\sim}{r}=\begin{pmatrix}-16\\\alpha\\10\end{pmatrix}+\mu\begin{pmatrix}1\\-4\\9\end{pmatrix}$$

where α is a constant and λ and μ are scalars.

(i) Given that the two lines intersect, find the value of α, and find the position vector of the point of intersection A. **3**

(ii) Show that the acute angle between l_1 and l_2 is $60°$. **1**

(iii) If B is a point on l_1 such that AB is 10 units, find the perpendicular distance from B to the line l_2. **1**

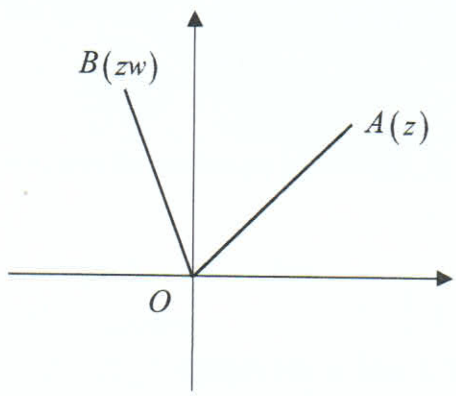
Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Find $\int x^2 \ln x \, dx$. **2**

(b) Using the substitution $t = \tan\frac{x}{2}$ or otherwise evaluate: **3**

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3\sin x - 4\cos x + 5} \, dx.$$

(c) As shown on the Argand diagram below, the complex numbers z and zw are represented by the points A and B respectively.



Given $z = re^{i\theta}$ and $w = e^{i\frac{\pi}{3}}$, where $r > 0$,

(i) Explain why OAB is an equilateral triangle. **2**

(ii) Write the complex number $z - zw$ in exponential form in terms of r and θ. **2**

Question 13 continues on Page 9

(d) A particle moves in a straight line. Its displacement x metres from the origin O after t seconds is given by

$$x = \cos 2t + \sin 2t + 1.$$

(i) Prove that the particle is moving in simple harmonic motion centred at $x = 1$. **2**

(ii) Find the first three times after $t = 0$ the particle is moving at a speed of 2 m/s. **4**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) Prove by contradiction that if $p, q > 0$ then $\frac{p}{q} + \frac{q}{p} \geq 2$. **3**

(b) A particle is moving along the x axis.

Its acceleration is given by $a = \frac{7-3x}{x^3}$, and the particle starts from rest at the point $x = 1$.

(i) Explain why the particle starts moving in the positive x direction. **1**

(ii) Let v be the velocity of the particle. Show that $v = \frac{\sqrt{x^2 + 6x - 7}}{x}$. **3**

(iii) Describe the motion of the particle after it passes $x = \frac{7}{3}$. **1**

Question 14 continues on Page 11

(c) Let $I_n = \int_0^1 \frac{x^{2n}}{1+x^2}\,dx$, where n is an integer and $n \geq 0$.

(i) Show that $I_n + I_{n-1} = \frac{1}{2n-1}$. **2**

(ii) Hence evaluate $\int_0^1 \frac{x^4}{1+x^2}\,dx$. **2**

(d) Use mathematical induction to prove that for all positive even integers n, **3**

$$4^{n+1} + 6^n \text{ is divisible by } 10.$$

End of Question 14

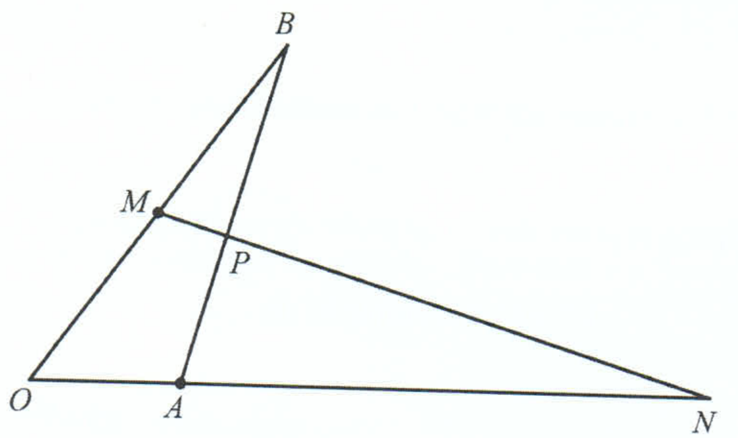
Question 15 (14 marks) Use a SEPARATE writing booklet

(a) (i) Explain why $\underset{\sim}{a} \cdot \underset{\sim}{b} \leq |\underset{\sim}{a}||\underset{\sim}{b}|$ for vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **1**

(ii) Use a suitable choice of three-dimensional vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ to prove that $3x+2y+6 \leq 7\sqrt{x^2+y^2+1}$ **1**

(iii) For what values of x and y does equality hold? **1**

(b) In the diagram below, ON, OB, AB and MN are all straight lines.



M is the midpoint of OB, and $OA : AN = 1:4$.

$\overrightarrow{OA} = 2\underset{\sim}{a}$ and $\overrightarrow{OB} = 2\underset{\sim}{b}$.

Let $\overrightarrow{AP} = \lambda\,\overrightarrow{AB}$, where λ is a scalar.

(i) Show that $\overrightarrow{OP} = 2(1-\lambda)\underset{\sim}{a} + 2\lambda\underset{\sim}{b}$. **1**

(ii) If $\overrightarrow{MP} = \mu\,\overrightarrow{MN}$, where μ is a scalar, find another expression for $\overrightarrow{OP}$ in terms of μ, $\underset{\sim}{a}$ and $\underset{\sim}{b}$. **2**

(iii) Hence find the ratio in which P divides AB. **2**

Question 15 continues on Page 13

(c) (i) Show that $\dfrac{1}{(k-1)k(k+1)} > \dfrac{1}{k^3}$ where $k > 1$. **2**

(ii) Given that $S_n = \dfrac{1}{3^3} + \dfrac{1}{4^3} + \dfrac{1}{5^3} + \ldots + \dfrac{1}{n^3} = \sum_{k=3}^{n} \dfrac{1}{k^3}$, **4**

by using partial fractions on part (i) or otherwise, prove that $S_n < \dfrac{1}{12}$ $\forall n \geq 3$.

End of Question 15

Question 16 (16 marks) Use a SEPARATE writing booklet

(a) (i) Prove that: **2**

$$\frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1} = i\tan\frac{\theta}{2}.$$

(ii) Find the roots of the equation $w^5 = 1$.

Leave your answers in polar form with principal arguments. **2**

(iii) Hence show that the roots of the equation $\left(\frac{2+z}{2-z}\right)^5 = 1$ are **3**

$$z = 2i\tan\left(\frac{k\pi}{5}\right) \text{ where } k = 0, \pm 1, \pm 2.$$

(b) (i) By finding the equation of the tangent to $y = \log_e x$ at $x = 1$, explain why **1**

$$\log_e x \leq x - 1 \text{ for } x > 0.$$

(ii) If $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ are positive numbers such that **3**
$a_1 + a_2 + a_3 + \ldots + a_n = 1$ and $a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n = 1$,

prove that $b_1^{a_1} b_2^{a_2} b_3^{a_3} \ldots b_n^{a_n} \leq 1$.

Question 16 continues on Page 15

(c) (i) Explain why $\int \left[f'(x)g(x) + g'(x)f(x) \right] dx = f(x)g(x) + C$. **1**

(ii) A function $f(x)$ satisfies the equation **4**

$$f'(x) - \frac{1}{x} f(x) = x^2 \cos 2x .$$

By first multiplying both sides of the equation by $\frac{1}{x}$, use the result in (i) to find $f(x)$ given that $f(\pi) = 2\pi$.

End of paper

2022 NSG Extension 2 Trial Solutions

Multiple Choice

Summary: 1. D 2. C 3. C 4. A 5. C
6. B 7. C 8. B 9. D 10. B

① $|\underset{\sim}{a}| = \sqrt{2^2+1^2+3^2}$
$= \sqrt{14}$

$\therefore \hat{\underset{\sim}{a}} = \frac{1}{\sqrt{14}}\begin{pmatrix}2\\1\\-3\end{pmatrix}$ (D)

② let P: you eat your meat, Q: You get pudding
Statement: $\neg P \Rightarrow \neg Q$
Contrapositive: $Q \Rightarrow P$
ie. if you had pudding then you ate your meat (C)

③ $z^4 - 1 = 0$
$(z^2-1)(z^2+1) = 0$
$z = \pm 1,\ z = \pm i$

$z^2 + 3iz - 2 = 0$
Check $z = -i$: $-1 + 3 - 2 = 0$ $\therefore$ root
Sum of roots: $-i + \alpha = -3i$
$\alpha = -2i$

Distinct roots: $\pm 1,\ \pm i,\ -2i$ ie 5 (C)

④ $|x-iy| = \sqrt{x^2+(-y)^2}$
$= \sqrt{x^2+y^2}$
$= 1$
$\therefore |(x-iy)^{-12}| = |(x+iy)^{12}|$

$\operatorname{Arg}(x-iy) = -\operatorname{Arg}(x+iy)$
$\operatorname{Arg}(x-iy)^{-12} = -12\operatorname{Arg}(x-iy)$
$= 12\operatorname{Arg}(x+iy)$
$= \operatorname{Arg}(x+iy)^{12}$

$\therefore (x-iy)^{-12} = (x+iy)^{12} = a+bi$ (A)

⑤ $n^2 = 2 \Rightarrow n = \sqrt{2} \Rightarrow$ period $= \frac{2\pi}{\sqrt{2}} = \pi\sqrt{2}$ seconds

Origin to extreme is $\frac{1}{4}$ cycle. $\Rightarrow \frac{\pi\sqrt{2}}{4}$ seconds (C)

(6) $\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix}1\\-2\\1\end{pmatrix}$

$\vec{AC} = \vec{OC} - \vec{OA} = \begin{pmatrix}2\\\mu-1\\\lambda\end{pmatrix} = 2\begin{pmatrix}1\\-2\\1\end{pmatrix}$ — to match the components

$\mu - 1 = -4 \Rightarrow \mu = -3$, $\lambda = 2$

(B)

(7) (C) is the only integrand which is true $\forall x \in [-\sqrt{3}, \sqrt{3}]$

(8) $z\bar{z} = |z|^2 = [Re(z)]^2 + [Im(z)]^2$

$z - \bar{z} = 2i\,Im(z) \Rightarrow (z-\bar{z})^2 = -4[Im(z)]^2$

$$\frac{4z\bar{z}}{(z-\bar{z})^2} = \frac{\cancel{4}[Re\,z]^2 + \cancel{4}[Im\,z]^2}{-\cancel{4}[Im\,z]^2} = -1 - \left(\frac{Re\,z}{Im\,z}\right)^2$$

(B)

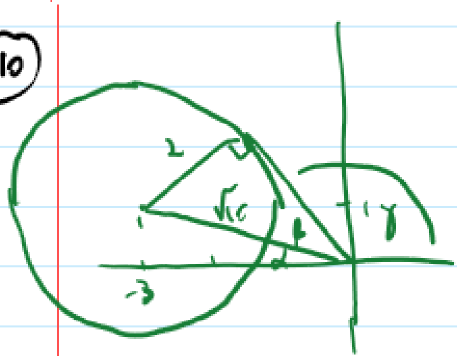
(9) Negation of a universal quantifier is an existential quantifier

A	B	$A \Rightarrow B$	$\neg(A \Rightarrow B)$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	T	F

which is the same as $A \wedge \neg B$

(D)

(10)



$\alpha = \tan^{-1}\left(\frac{1}{3}\right)$

$\beta = \sin^{-1}\left(\frac{2}{\sqrt{10}}\right)$

$\gamma = \pi - \tan^{-1}\left(\frac{1}{3}\right) - \sin^{-1}\left(\frac{2}{\sqrt{10}}\right)$

$= \pi + \tan^{-1}\left(-\frac{1}{3}\right) - \sin^{-1}\left(\frac{2}{\sqrt{10}}\right)$

(B)

Question 11

(a) (i) $z + w = 3 - 3i$

$= \boxed{3\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)}$ [2]

Please note that $\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}$ is not mod-arg form.

(ii) $\frac{z}{w} = \frac{1+3i}{2-6i} \times \frac{2+6i}{2+6i}$

$= \frac{-16+12i}{40}$

$= \boxed{-\frac{2}{5} + \frac{3}{10}i}$ [2]

Each error here was penalised one mark. Please take extra care on the early questions to avoid careless errors as these can be costly.

(b) $x^2 + 4x + 4 + y^2 - 6y + 9 + z^2 = 12 + 4 + 9$

$(x+2)^2 + (y-3)^2 + z^2 = 25$

sphere, radius 5, centre $(-2, 3, 0)$

$\therefore \boxed{\left|\underset{\sim}{r} - \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix}\right| = 5}$ where $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ [2]

Students were able to write the equation in centre-radius form. Some issues occurred when writing the vector equation with students using column vector notation rather than $|\ |$ for modulus. Others used z in their solution without clearly indicating it was a vector. This was problematic because there was already a z in the question.

(c) $\int \frac{x^2 - 2x}{x^2 - 2x + 10}\,dx = \int \frac{x^2 - 2x - 10 + 10}{x^2 - 2x + 10}\,dx$

$= \int \left(1 + \frac{10}{(x-1)^2 + 9}\right) dx$

$= \boxed{x + \frac{10}{3}\tan^{-1}\frac{x-1}{3} + c}$ [2]

Well done.

(d) RTP: $4^n > 3n+7 \quad \forall n \in \mathbb{Z}^+$

Test $n=2$: $4^2 = 16 \qquad 3(2)+7 = 13$

$4^2 > 3(2)+7$

$\therefore$ true for $n=2$

Assume true for $n=k$:

ie. $4^k > 3k+7$

Prove true for $n=k+1$:

$$\begin{aligned} 4^{k+1} - 3(k+1) - 7 &= 4(4^k) - 3k - 10 \\ &> 4(3k+7) - 3k - 10 \quad \text{(by assumption)} \\ &= 9k + 18 \\ &> 0 \quad \forall k > 1 \end{aligned}$$

$\therefore 4^{k+1} > 3(k+1) + 7$

$\therefore$ by MI, $4^n > 3n+7 \quad \forall n > 1$. [3]

Lots of small algebraic errors here which were penalised because in a proof the result you are trying to show is already known. This means you must clearly explain your way to the result showing all the steps.

Students were penalised ½ mark if they didn't say 'by assumption' or similar when completing the inductive step.

Students could have done a better job justifying how they got from second last line to the last line of their proof by stating that $k > 1$ or similar.

(e) From $v^2 = n^2(a^2 - x^2)$, max $v = na$ (when $x = 0$)

$na = 4 \quad -(1)$

From $\ddot{x} = -n^2 x$, max acc $= n^2 a$ (when $x = -a$)

$n^2 a = 12 \quad -(2)$

$(2) \div (1)$: $n = 3 \Rightarrow$ period $= \frac{2\pi}{3}$

from (1): amp $= \frac{4}{3}$ [3]

OR

$x = a \sin nt$

$\dot{x} = na \cos nt \Rightarrow |\dot{x}_{max}| = na$

$\ddot{x} = -n^2 a \sin nt \Rightarrow |\ddot{x}_{max}| = n^2 a$

then as before

Need to start your solution using a known result for SHM and then developing the equations. So either:

$v^2 = n^2(a^2 - x^2)$ and $\ddot{x} = -n^2 x$

OR

$x = a \sin nt$, $\dot{x} = an \cos nt$, $\ddot{x} = -an^2 \sin nt$

Please note that you cannot start by quoting max $v = na$ and max $acc = n^2 a$.

The majority of students incorrectly stated that max acceleration occurs at $x = a$ rather than at minimum displacement i.e $x = -a$. This meant that they had issues dealing with the negatives in their solution and many had to end up fudging the result or just ignore the negative. This was penalised.

Question 12

(a)

Answer is solid line

[2]

Generally well done. Learn the standard forms – this is not an interval or an arc.
Make sure the two rays appear to lie on the same line, preferably by dashing the interval between.

(b) $\int_{\pi/4}^{\pi/3} \sec^4 x \tan^2 x \, dx = \int_{\pi/4}^{\pi/3} \sec^2 x \cdot (1+\tan^2 x)\tan^2 x \, dx$

$= \int_{\pi/4}^{\pi/3} (\tan^2 x + \tan^4 x) \cdot d(\tan x)$

$= \left[\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x\right]_{\pi/4}^{\pi/3}$

$= \left(\sqrt{3} + \frac{9}{5}\sqrt{3}\right) - \left(\frac{1}{3} + \frac{1}{5}\right)$

$= \boxed{\frac{14}{5}\sqrt{3} - \frac{8}{15}}$

[3]

Mostly well done. Some wasted time doing IBP.

(c) (i) If x is even then $x^2 - 6x + 5$ is odd [1]

Well done. But why say ‘not odd’ when you can say ‘even’.

(ii) let x be even, ie. $x = 2n$, $n \in \mathbb{Z}$

$x^2 - 6x + 5 = (2n)^2 - 6(2n) + 5$

$= 2(2n^2 - 6n + 2) + 1$

$= 2M + 1$ where $M = 2n^2 - 6n + 2 \in \mathbb{Z}$ as $n \in \mathbb{Z}$

$\therefore$ if x is even then $x^2 - 6x + 5$ is odd

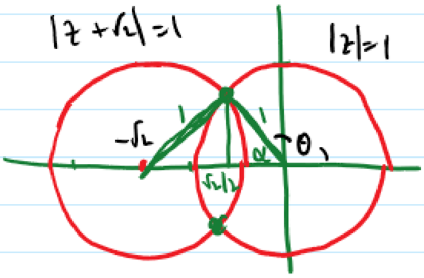
by contraposition, if $x^2 - 6x + 5$ is even then x is odd

[2]

Well done. Make sure you indicate that $n \in \mathbb{Z}$.
Don’t just conclude with the contrapositive. Refer back the the original statement to be proved.

(d) $|e^{i\theta}| = 1$

ie. $|z| = 1$



$\cos\alpha = \frac{\sqrt{2}}{2}$

$\alpha = \pi/4$

$\therefore \theta = \frac{3\pi}{4}$

$\boxed{z = e^{\pm i \frac{3\pi}{4}}}$ [3]

Numerous errors arose here. Those who drew a correct diagram generally had an easy 3 marks.

$\left|z+\sqrt{2}\right| = 1$ does NOT mean that $z+\sqrt{2} = \pm 1$.

You can't evaluate $\left|e^{i}+\sqrt{2}\right|$ without first collecting real and imaginary parts.

(e) (i) $9 - 8\lambda = -16 + \mu$

$8\lambda + \mu = 25$

$4 + 5\lambda = 10 + 9\mu$

$5\lambda - 9\mu = 6$

Solving simultaneously: $\lambda = 3$, $\mu = 1$

$2 - 3\lambda = \alpha + 4\mu$

$-7 = \alpha + 4$

$\alpha = -11$

$\vec{OA} = \begin{pmatrix} 9 \\ 2 \\ 4 \end{pmatrix} + 3\begin{pmatrix} -8 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -15 \\ -7 \\ 19 \end{pmatrix}$ [3]

Well done. A point is not a position vector.

(ii) if $\underset{\sim}{u} = \begin{pmatrix} -8 \\ -3 \\ 5 \end{pmatrix}$, $\underset{\sim}{v} = \begin{pmatrix} 1 \\ -4 \\ 9 \end{pmatrix}$ then $\underset{\sim}{u} \cdot \underset{\sim}{v} = -8 + 12 + 45 = 49$

$|\underset{\sim}{u}| = 7\sqrt{2}$, $\underset{\sim}{v} = \sqrt{53}$

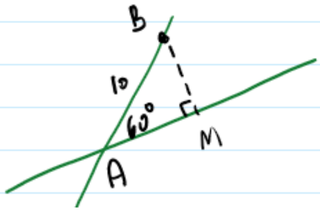
$\cos\theta = \frac{49}{7\sqrt{2} \cdot 7\sqrt{2}} = \frac{1}{2}$

$\theta = 60°$ [1]

It is a SHOW question. Don't skip any steps.

Direction vectors were already present in the equations of the lines. There was no need to subtract vectors to get direction vectors.

(iii)



$BM = 10\sin 60°$

$= \boxed{5\sqrt{3} \text{ units}}$ [1]

Many students treated B as the tip of a projection onto l_1, instead of projecting B onto l_2. Students who made the correct interpretation and drew a diagram generally found this easy.

Question 13

(a) $\int x^2 \ln x \, dx = \int \ln x \cdot d\left(\frac{x^3}{3}\right)$

$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \cdot d(\ln x)$

$= \frac{x^3}{3}\ln x - \int \frac{x^3}{3} \cdot \frac{1}{x}\, dx$

$= \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\, dx$

$= \boxed{\frac{x^3}{3}\ln x - \frac{x^3}{9} + c}$ [2]

$= \frac{x^3}{9}(3\ln x - 1) + c$

Done very well.

(b) $\int_{\pi/3}^{\pi/2} \frac{dx}{3\sin x - 4\cos x + 5}$

let $t = \tan\frac{x}{2}$

$x = 2\tan^{-1} t$

$dx = \frac{2dt}{1+t^2}$

$x = \frac{\pi}{3}, \; t = \frac{1}{\sqrt{3}}$

$x = \frac{\pi}{2}, \; t = 1$

$= \int_{1/\sqrt{3}}^{1} \frac{\frac{2dt}{1+t^2}}{\frac{6t}{1+t^2} - \frac{4(1-t^2)}{1+t^2} + 5}$

$= \int_{1/\sqrt{3}}^{1} \frac{2dt}{6t - 4 + 4t^2 + 5 + 5t^2}$

$= \int_{1/\sqrt{3}}^{1} \frac{2dt}{9t^2 + 6t + 1}$

$= \int_{1/\sqrt{3}}^{1} 2(3t+1)^{-2}\, dt$

$= \frac{2}{-1\times 3}\left[(3t+1)^{-1}\right]_{1/\sqrt{3}}^{1}$

$= -\frac{2}{3}\left[\frac{1}{3t+1}\right]_{1/\sqrt{3}}^{1}$

$= -\frac{2}{3}\left(\frac{1}{4} - \frac{1}{\sqrt{3}+1}\right)$

$= \frac{2}{3} \cdot \frac{4 - (\sqrt{3}+1)}{4(\sqrt{3}+1)}$

$= \boxed{\frac{3-\sqrt{3}}{6(1+\sqrt{3})}}$ [3]

Also done well overall. Some students made silly mistakes in their algebra when simplifying the fraction.

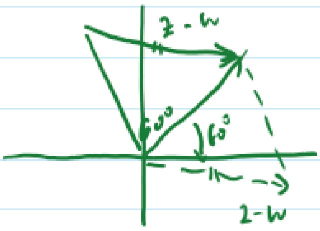
(c) (i) $|w| = 1$ and $\text{Arg}\, w = \frac{\pi}{3} = 60°$

$\therefore |zw| = |z|$ and $\text{Arg}(zw) = \text{Arg}\, z + \frac{\pi}{3}$

$\therefore \Delta OAB$ is equilateral (2 sides equal, included angle $60°$) [2]

Not done well. Many students did not sufficiently "explain" or show why OAB was an equilateral triangle. Students needed to show that they found the modulus of z and zw and that they were equal rather than simply stating that OB=OA without any explanation. Also, ideally students would find the rest of the angles other than AOB to show that it was equilateral (although this was not penalised this time).

(ii)



$$z - zw = z \cdot e^{-i\pi/3}$$
$$= re^{i\theta} \cdot e^{-i\pi/3}$$
$$= \boxed{re^{i(\theta - \pi/3)}}$$

[2]

OR
$$z - zw = z(1-w)$$
$$= re^{i\theta}(1 - e^{i\pi/3})$$
$$= re^{i\theta}\left[1 - \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]$$
$$= re^{i\theta}\left(\frac{1}{2} - i\sin\frac{\pi}{3}\right)$$
$$= re^{i\theta}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$
$$= re^{i\theta} \cdot e^{-i\pi/3}$$

Also not done well. The easier way to do this was geometrically which most students attempted to do, recognising that vector BA was BO rotated anti-clockwise by pi/3. (Some students are still getting confused with the vector representation of a complex number and were stating the z-zw was vector AB.). However, doing it this way gives -re^(itheta + i 2pi/3) which is not in exponential form since there is a negative in front of r.

(d) (i)
$$x = \cos 2t + \sin 2t + 1$$
$$\dot{x} = -2\sin 2t + 2\cos 2t$$
$$\ddot{x} = -4\cos 2t - 4\sin 2t$$
$$= -4(\cos 2t + \sin 2t)$$
$$\ddot{x} = -4(x-1)$$
$\therefore$ SHM about $x = 1$

[2]

Done very well.

(ii) $\pm 2 = -2\sin 2t + 2\cos 2t$

$\cos 2t - \sin 2t = \pm 1$

Let $\cos 2t - \sin 2t = R\cos(2t + \alpha)$

$= R(\cos 2t \cos\alpha - \sin 2t \sin\alpha)$

$= (R\cos\alpha)\cos 2t - (R\sin\alpha)\sin 2t$

$\therefore R\cos\alpha = 1$ ①, $R\sin\alpha = 1$ ②

①^2 + ②^2 : $R^2(\cos^2\alpha + \sin^2\alpha) = 1^2 + 1^2$

$R = \sqrt{2}$

$\cos\alpha = \frac{1}{\sqrt{2}}$

$\alpha = \frac{\pi}{4}$ (1st quadrant)

$\sqrt{2}\cos(2t + \frac{\pi}{4}) = \pm 1$

$\cos(2t + \frac{\pi}{4}) = \pm 1/\sqrt{2}$

$2t + \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \ldots$

$2t = \cancel{0}, \frac{\pi}{2}, \pi, 3\pi/2$

$t = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$

Majority of students didn't realise the question was giving speed not velocity and so did not realise they had to find solutions for both +2 and -2. Other minor mistakes were made when converting the expression to auxiliary form or when solving the equation.

Question 14

(a) RTP: $\forall p,q > 0, \quad \frac{p}{q} + \frac{q}{p} \geq 2$

Assume: $\exists p,q > 0, \quad \frac{p}{q} + \frac{q}{p} < 2$

$$\therefore \frac{p^2+q^2}{pq} < 2$$
$$p^2+q^2 < 2pq$$
$$p^2+q^2-2pq < 0$$
$$(p-q)^2 < 0$$

Contradiction, as a perfect square can't be -ve

$\therefore \neg\left(\exists p,q > 0, \frac{p}{q} + \frac{q}{p} < 2\right)$

ie. $\forall p,q > 0, \quad \frac{p}{q} + \frac{q}{p} \geq 2$ [3]

Students who tried to combine a proof by contradiction with a direct proof were not awarded full marks. The assumption must lead you straight to a statement which contradicts the assumption, is self-contradictory, or (in this question) contradicts "common sense".

You do not negate a statement by negating the domain of the statement. The negation of "P is true for all a in A" is "There exists an a in A for which P is not true".

(b) (i) $t=0, x=1, \dot{x}=0$ (given), $\ddot{x} = \frac{7-3}{1^3} > 0$

Starting from rest and accelerating to right

$\therefore$ initially moves in +ve direction [1]

Not well done. Most students recognised that the initial acceleration is positive, but the direction of acceleration is not the direction of travel. You needed to mention that the initial velocity is zero.

(ii) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{7-3x}{x^3} = 7x^{-3} - 3x^{-2}$

$$\left[\tfrac{1}{2}v^2\right]_0^v = \int_1^x \left(7x^{-3} - 3x^{-2}\right) dx$$

$$\left[v^2\right]_0^v = 2\left[\frac{3}{x} - \frac{7}{2x^2}\right]_1^x$$

$$v^2 = 2\left(\frac{3}{x} - \frac{7}{2x^2} - 3 + \frac{7}{2}\right)$$

$$= \frac{6}{x} - \frac{7}{x^2} + 1$$

$$= \frac{x^2+6x-7}{x^2}$$

$$v = \pm\frac{\sqrt{x^2+6x-7}}{x}$$

but $x^2+6x-7=0 \Rightarrow x=-7,1$, so once particle move to right of $x=1$ it can't stop

$\therefore v > 0 \ \forall t > 0, \ \therefore v = \frac{\sqrt{x^2+6x-7}}{x}$ [3]

Most students could perform the derivation, but only a handful could properly justify the choice of the positive case. The fact that the initial direction of motion is in the positive direction is not sufficient to claim that the velocity will always be positive. Consider simple harmonic motion for a counter-example – the velocity can be positive or negative at a single value of x. See solution.

(iii) (I) $v > 0$ for all time, and as $x \to \infty$, $v \to 1^-$

(II) if $x > \frac{2}{3}$, $a < 0$

(III) when $x = 7/3$, $v = \frac{4}{\sqrt{3}}$

$\therefore$ particle moves forever to the right, slowing from $v = \frac{4}{\sqrt{3}}$ to a limiting velocity of 1m/s. [1]

Not done well. Any student that suggested that the particle stops, changes direction, or moves at a constant velocity was not awarded the mark. The limiting velocity was needed to get the full mark.

(c) (i)
$$\begin{aligned}
I_n + I_{n-1} &= \int_0^1 \frac{x^{2n}}{1+x^2}\,dx + \int_0^1 \frac{x^{2n-2}}{1+x^2}\,dx \\
&= \int_0^1 \left(\frac{x^{2n}}{1+x^2} + \frac{x^{2n-2}}{1+x^2}\right)dx \\
&= \int_0^1 \frac{x^{2n-2}(x^2+1)}{1+x^2}\,dx \\
&= \int_0^1 x^{2n-2}\,dx \\
&= \frac{1}{2n-1}\left[x^{2n-1}\right]_0^1 \\
&= \frac{1}{2n-1}
\end{aligned}$$
[2]

This was an easy two marks for most who followed the method in the solutions, and led nowhere for those who attempted IBP.

(ii)
$$\begin{aligned}
\int_0^1 \frac{x^4}{1+x^2}\,dx &= I_2 \\
&= \frac{1}{3} - I_1 \quad \text{(from (i))} \\
&= \frac{1}{3} - (1 - I_0) \quad \text{(from (i))} \\
&= -\frac{2}{3} + \int_0^1 \frac{dx}{1+x^2} \\
&= -\frac{2}{3} + \left[\tan^{-1}x\right]_0^1 \\
&= \boxed{\frac{\pi}{4} - \frac{2}{3}}
\end{aligned}$$
[2]

A large number of students interpreted this integral as I_4 instead of I_2.

(d) RTP: $10 \mid (4^{n+1} + 6^n) \quad \forall n = 2r, \ r \in \mathbb{Z}^+$

Test $n = 2$: $4^3 + 6^2 = 100 = 10(10)$

$\therefore$ true for $n = 2$

Assume true for $n = k$: $4^{k+1} + 6^k = 10M, \ M \in \mathbb{Z}$

Prove true for $n = k+2$: $4^{k+3} + 6^{k+2} = 16(4^{k+1}) + 36(6^k)$

$= 16(4^{k+1}) + 36(10M - 4^{k+1})$ (assumption)

$= 360M - 20(4^{k+1})$

$= 10(36M - 2(4^{k+1}))$

$= 10N, \quad N = 36M - 2(4^{k+1}) \in \mathbb{Z}$

$\therefore$ true for $n = k+2$ when true for $n = k$

$\therefore$ By MI, $10 \mid (4^{n+1} + 6^n)$

$\forall n = 2r, \ r \in \mathbb{Z}^+$

[3]

There is only need to substitute for one of the exponential terms to get the result. Students who tried to substitute for both got themselves into a mess.

Please use notation correctly. $3 \mid 12$ (3 divides 12), not $12 \mid 3$.

Question 15

(a) (i) $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$

$\leq |\underset{\sim}{a}||\underset{\sim}{b}|$ (as $\cos\theta \leq 1$) [1]

(with equality when $\cos\theta = 1$, ie. $\theta = 0°$)

Poorly done. Many students were unsure where to start this relatively simple proof. See solutions.

(ii) let $\underset{\sim}{a} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$, $\underset{\sim}{b} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 3x + 2y + 6$

$|\underset{\sim}{a}| = \sqrt{x^2 + y^2 + 1}$, $|\underset{\sim}{b}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$

$\therefore |3x + 2y + 6| \leq 7\sqrt{x^2 + y^2 + 1}$ [1]

Well done.

(iii) When the vectors are parallel.

ie. $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$

$\therefore \lambda = \frac{1}{6} \Rightarrow \boxed{x = \frac{1}{2},\ y = \frac{1}{3}}$ [1]

Poorly done. Many students thought about part (ii) as an expression in x and y and tried to work out the answer algebraically. Here, you had to relate the expression to vectors as the question was originally framed in the previous parts.

(b) (i) $\vec{OP} = \vec{OA} + \vec{AP}$

$= 2\underset{\sim}{a} + \lambda\vec{AB}$

$= 2\underset{\sim}{a} + \lambda(\vec{OB} - \vec{OA})$

$= 2\underset{\sim}{a} + \lambda(2\underset{\sim}{b} - 2\underset{\sim}{a})$

$= 2\underset{\sim}{a} + 2\lambda\underset{\sim}{b} - 2\lambda\underset{\sim}{a}$

$= 2(1-\lambda)\underset{\sim}{a} + 2\lambda\underset{\sim}{b}$ [1]

Well done.

(ii) $\vec{OP} = \vec{OM} + \vec{MP}$

$= \frac{1}{2}\vec{OB} + \mu\vec{MN}$

$= \underset{\sim}{b} + \mu(\vec{ON} - \vec{OM})$

$= \underset{\sim}{b} + \mu(5\vec{OA} - \frac{1}{2}\vec{OB})$

$= \underset{\sim}{b} + 10\mu\underset{\sim}{a} - \mu\underset{\sim}{b}$

$= \boxed{10\mu\underset{\sim}{a} + (1-\mu)\underset{\sim}{b}}$ [2]

Generally well done. Some students did not get the ratio *OA:ON* correct. If *OA:AN* is 1:4, then *OA* is 1/5 of *ON*.

(iii) $2(1-\lambda)\underset{\sim}{a} + 2\lambda\underset{\sim}{b} = 10\mu\underset{\sim}{a} + (1-\mu)\underset{\sim}{b}$

As $\underset{\sim}{a}, \underset{\sim}{b}$ independent (not parallel),

$$2(1-\lambda) = 10\mu \quad , \quad 2\lambda = 1-\mu$$
$$\lambda + 5\mu = 1 \; ① \qquad 2\lambda + \mu = 1 \; ②$$

$2\times① - ②: \quad 9\mu = 1 \quad \Rightarrow \mu = \frac{1}{9}$

Sub in ②: $\quad 2\lambda + \frac{1}{9} = 1 \quad \Rightarrow \lambda = \frac{4}{9}$

ie. $\overrightarrow{AP} = \frac{4}{9}\overrightarrow{AB}$

ie. P divides $\overrightarrow{AB}$ in ratio $\boxed{4:5}$ [2]

Mostly well done. Some careless mistakes in relatively long working. Several more students did not get the ratio correct at the end, if *AP* is 4/9 *AB*, then *AP:PB* is 4:5. A quick revision of ratio and fractions may be in order, most students can not afford to be losing marks on the simple parts of questions.

(c) (i) $\frac{1}{(k-1)k(k+1)} - \frac{1}{k^3} = \frac{k^2 - (k-1)(k+1)}{k^3(k-1)(k+1)}$

$$= \frac{1}{k^3(k-1)(k+1)}$$
$$> 0 \quad \text{as } k > 1$$

$\therefore \frac{1}{(k-1)k(k+1)} > \frac{1}{k^3} \quad \forall k > 1$ [2]

Done well. This is a show that question, make sure you are not skipping lines. For instance, the start and end points have to match the question. Also, some lines require justification. Solutions either started with LHS – RHS, or built up the expression. To get full marks the second way, you needed a comparison of $k^3 - k$ and *k,* not just their reciprocals.

(ii) let $\frac{1}{(k-1)k(k+1)} = \frac{A}{k-1} + \frac{B}{k} + \frac{C}{k+1}$

$$1 = Ak(k+1) + B(k-1)(k+1) + Ck(k-1)$$

$k=-1: \quad 1 = 2C \quad \Rightarrow C = \frac{1}{2}$

$k=0: \quad 1 = -B \quad \Rightarrow B = -1$

$k=1: \quad 1 = 2A \quad \Rightarrow A = \frac{1}{2} \qquad (*)$

$\therefore S_n = \sum_{k=3}^{n} \frac{1}{k^3}$

$< \sum_{k=3}^{n}\left(\frac{1/2}{k-1} - \frac{1}{k} + \frac{1/2}{k+1}\right)$ from (i) & (*)

$= \frac{1}{2}\sum_{2}^{n-1}\frac{1}{k} - \sum_{k=3}^{n}\frac{1}{k} + \frac{1}{2}\sum_{k=4}^{n+1}\frac{1}{k}$

$= \left[\frac{1}{2}\cancel{\sum_{k=4}^{n-1}}\frac{1}{k} + \frac{1}{2} + \frac{1}{3}\right] - \left[\cancel{\sum_{k=4}^{n-1}}\frac{1}{k} + \frac{1}{3} + \frac{1}{n}\right] + \frac{1}{2}\left[\cancel{\sum_{k=4}^{n-1}}\frac{1}{k} + \frac{1}{n} + \frac{1}{n+1}\right]$

$= \frac{1}{4} + \frac{1}{6} - \frac{1}{3} - \frac{1}{n} + \frac{1}{2n} + \frac{1}{2(n+1)}$

$= \frac{1}{12} - \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+1}\right)$

$< \frac{1}{12} \quad$ as $\frac{1}{n} > \frac{1}{n+1}$

[4]

Partial fractions were done well and many students could then use the results together. Many students lost marks setting up or performing the required cancellation. Plan your setting out when working with complicated sums, so you can systematically check what is being combined across terms.

Question 16

(a) Let $t = \tan\frac{x}{2}$

$$LHS = \frac{\frac{1-t^2}{1+t^2} + \frac{2it}{1+t^2} - 1}{\frac{1-t^2}{1+t^2} + \frac{2it}{1+t^2} + 1} \times \frac{1+t^2}{1+t^2}$$

$$= \frac{1-t^2+2it-(1+t^2)}{1-t^2+2it+(1+t^2)} \div \frac{2}{2}$$

$$= \frac{-t^2+it}{1+it} \times \frac{1-it}{1-it}$$

$$= \frac{\cancel{-t^2}+it^3+it+\cancel{t^2}}{1+t^2}$$

$$= \frac{it\cancel{(1+t^2)}}{\cancel{1+t^2}} = i\tan\frac{\theta}{2}$$

Variety of methods used here. You are proving an identity so please don't skip steps and avoid just quoting results. In particular, students who used exponential form needed to explain how $e^{i\frac{\theta}{2}} + e^{-i\frac{\theta}{2}} = 2\cos\frac{\theta}{2}$ and $e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}} = 2\sin\frac{\theta}{2}$. The better responses did this by converting back to mod-arg form.
Many students used t-formulae and made some progress but could not complete the proof because they didn't 'realise' the denominator or correctly factorise the numerator to match the RHS.

(ii) $w^5 = 1 = \cos 2k\pi + i\sin 2k\pi, \quad k \in \mathbb{Z}$

$w = \cos\frac{2k}{5}\pi + i\sin\frac{2k}{5}\pi$

$w = \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right), \ \cos\left(-\frac{2\pi}{5}\right) + i\sin\frac{2\pi}{5}, \ \cos 0 + i\sin 0,$
$\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}, \ \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}.$

Well done, although the setting out could generally be improved. This could have been due to the time constraints at the end of the paper.

(iii) $\left(\frac{2+z}{2-z}\right)^5 = 1 \Rightarrow \frac{2+z}{2-z} = \text{cis}\frac{2k\pi}{5}, \ k = 0, \pm1, \pm2$ from (ii)

If $\frac{2+z}{2-z} = m$ then $2+z = m(2-z)$

$2+z = 2m - mz$

$z(1+m) = 2(m-1)$

$z = \frac{2(m-1)}{m+1}$

$\therefore$ roots take form: $z = 2\,\frac{\text{cis}\frac{2k\pi}{5} - 1}{\text{cis}\frac{2k\pi}{5} + 1} = 2i\tan\frac{k\pi}{5}$ from (i)

[2]

Generally well done with the main errors coming when students failed to realise that in order to solve for z, they first needed to set $\frac{2+z}{2-z} = \text{cis}\frac{2k\pi}{5}$ and then make z the subject.

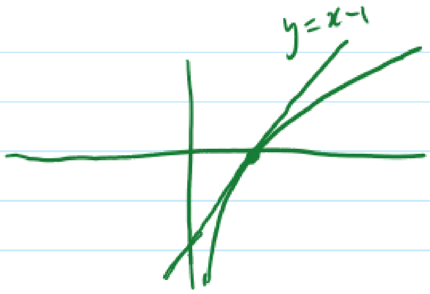
(b) (i) $y = \ln x$

$y' = \frac{1}{x}$

$x = 1,\ m_T = 1$

$y - \ln 1 = 1(x-1)$

tangent: $y = x - 1$



As $y = \ln x$ is concave down $\forall x$, it never lies above any tangent, $\therefore \ln x \le x - 1\ \forall x$ [1]

A fairly straightforward question but not well done. Please ensure your solution follows any instructions given in the question. (In this case 'by finding the equation of the tangent'). Please also take care with a written explanation to ensure you are using mathematical notation and terms correctly. Some students referred to logx having max. gradient at $x=1$, ignoring the gradient when $x<1$. If you are trying to explain 'from the graph' then please make sure your graph is clear.

(ii) $\ln(b_1^{a_1} b_2^{a_2} \dots b_n^{a_n}) = \ln(b_1^{a_1}) + \ln(b_2^{a_2}) + \dots + \ln(b_n^{a_n})$

$= a_1 \ln b_1 + a_2 \ln b_2 + \dots + a_n \ln b_n$

$\le a_1(b_1 - 1) + a_2(b_2 - 1) + \dots + a_n(b_n - 1)$ (from i)

$= a_1 b_1 + a_2 b_2 + \dots + a_n b_n - (a_1 + a_2 + \dots + a_n)$

$= 1 - 1$

$= 0$

$b_1^{a_1} b_2^{a_2} \dots b_n^{a_n} \le e^0 = 1$ [3]

Students had difficulty with this question. If you are not sure where to start then try linking back to the result you proved in the previous part. In this case that meant taking the log of the LHS of the inequality. Many students had a number of attempts but marks were only awarded if your response made at least some progress towards the solution.

(c) (i) $\frac{d}{dx} f(x)g(x) = f'(x)g(x) + g'(x)f(x)$ (product rule)

$\therefore \int [f'(x)g(x) + g'(x)f(x)]\,dx = f(x)g(x) + c$ [1]

Mostly well done, although it would have been nice to see students explain that they were using the Product Rule (rather than the Chain Rule!).
Some students used IBP to explain the result, and were generally successful, but this was a more time-consuming method.

(ii) $f'(x) - \frac{1}{x} f(x) = x^2 \cos 2x$

$\frac{1}{x} f'(x) - \frac{1}{x^2} f(x) = x \cos 2x$

$\frac{1}{x} f'(x) + \frac{d}{dx}\left(\frac{1}{x}\right) f(x) = x \cos 2x$

$\frac{d}{dx}\left(\frac{1}{x} f(x)\right) = x \cos 2x \qquad$ (part (i))

(*) $\frac{1}{x} f(x) = \int x \cos 2x \, dx$

$= \int x \cdot d\left(\frac{1}{2}\sin 2x\right)$

$= \frac{1}{2} x \sin 2x - \int \frac{1}{2} \sin 2x \cdot d(x)$

$\frac{1}{x} f(x) = \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + c$

$f(\pi) = 2\pi$: $\frac{1}{\pi} \cdot 2\pi = \frac{1}{2}\pi \sin 2\pi + \frac{1}{4} \cos 2\pi + c$

$2 = 0 + \frac{1}{4} + c$

$c = \frac{7}{4}$

$\therefore \frac{1}{x} f(x) = \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + \frac{7}{4}$

$f(x) = \frac{1}{2} x^2 \sin 2x + \frac{1}{4} x \cos 2x + \frac{7x}{4}$

OR (*)

$\left[\frac{1}{x} f(x)\right]_{\pi}^{x} = \int_{\pi}^{x} x \cos 2x \, dx$

$\frac{1}{x} f(x) - \frac{1}{\pi} f(\pi) = \frac{1}{2} \int_{\pi}^{x} x \cdot d(\sin 2x)$

$\frac{1}{x} f(x) - \frac{1}{\pi} \cdot 2\pi = \frac{1}{2}\left[x \sin 2x\right]_{\pi}^{x} - \frac{1}{2} \int_{\pi}^{x} \sin 2x \cdot d(x)$

$\frac{1}{x} f(x) - 2 = \frac{1}{2} x \sin 2x - 0 + \frac{1}{4}\left[\cos 2x\right]_{\pi}^{x}$

$\frac{1}{x} f(x) = \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x - \frac{1}{4} + 2$

$= \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + \frac{7}{4}$

$f(x) = \frac{1}{2} x^2 \sin 2x + \frac{1}{4} x \cos 2x + \frac{7x}{4}$

The main errors here were not correctly linking with the result in part (i) and not realising that the constant also needed to be multiplied by x.